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Stochastic modelling of rail fatigue

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Résumé — The dynamical response of a train rolling on a real track depends on several parameters. Most of them cannot be accurately identified and have to be considered as uncertain implying that the wheel-rail contact parameters cannot be completely characterized using deterministic models. The aim of this paper is the construction of a probabilistic model of the rail fatigue life considering the track geometry and the rail wear as random fields indexed by the space (curvilinear abscissa of the track) and statistically representative of the considered railway network.
Mots clés — rail fatigue, random field modelling, polynomial chaos metamodelling.

1 Introduction

The aim of this work is to construct a stochastic modelling of the rail fatigue life by means of numerical simulations which take the track geometry and the rail wear stochastic fields as input parameters.

The features of input random fields are identified using measurement data. Using the Karhunen-Loève expansion, a random field can be approximated by its truncated projection on an orthogonal basis. In this way it is expressed as a sum of a finite set of deterministic spatial functions multiplied by random decorrelated coefficients. Then, the multivariate distributions of the random projection coefficients of the basis are characterized using the Polynomial Chaos Expansion (PCE). This consists in expanding the random coefficients in a sum of deterministic coefficients multiplied by a polynomial basis of a multivariate random variable (called germ) whose probability density function is known. The PCE deterministic coefficients are identified via log-likelihood maximization. In this way, by generating the polynomial chaos germs, it is possible to obtain independent realisations of the field [1]. The curve radius, the rail age and the train operational velocity introduce non-stationary effects that have to be taken into account to model the track.

The tracks thus generated are introduced as the input of a railway dynamics software (*SIMPACK*) to simulate the passage of the vehicle on a curved track with geometric irregularities. The rail-wheel contact forces and the surface stresses fields are computed in each location of the spatial domain. A semi-Hertzian contact model taking into account the rail plasticity [2] is used in the simulations. The statistics of the parameters on which the contact patch depends are obtained by Monte Carlo simulations.

Using the surface stresses calculated in the previous step as boundary condition, a finite element calculation allows to compute the stresses inside the rail in a fixed location of the spatial domain. After repeated passages of the train the plastic strain field is stabilized, i.e. it does not vary after another passage. Using the algorithm proposed in [3] the stresses corresponding to this steady state are obtained and a fatigue criterion is applied on them. Since this step is computationally very expensive (more than one hour per simulation), a PCE-based meta-modelling technique [4] is employed to estimate the fatigue index in the complete spatial domain.

2 Track irregularities and wear random model

The available data from the measurement campaigns, concerning the RER line A, are processed in order to obtain realizations of the fields indexed by the curvilinear abscissa, s.

Concerning the track geometry irregularities four quantities are measured :
— Gauge, $G(s)$, the spacing of the rails measured between the inner faces
— Superelevation, $E(s)$, the difference in height between the two rails

— Horizontal curvature, $C_H(s)$, referred to the track median line
— Vertical curvature, $C_V(s)$, referred to the track median line

The irregularities are defined as the differences between the measurements and the corresponding design (or theoretical) known values. The irregularities fields thus defined are centred and decorrelated from the horizontal layout of the track.

Another set of measurements concerns the rail profiles. These measurements are treated and compared to a theoretical rail profile to obtain the wear field. This latter is modelled as a random field indexed by the rail and the track curvilinear abscissa.

The main steps of the random fields modelling are here reported.

Let $(\Theta, \mathcal{F}, P)$ be a probability space. Using a Karhunen-Loève expansion (KLE, [5]), a random field $f(s,\theta)$ (that could be any irregularities or the wear), with $\theta \in \Theta$, is approximated by its truncated projection on an orthogonal basis. In this way it is expressed as a sum of M deterministic spatial functions $\Phi_i(s)$ multiplied by random decorrelated and normalized coefficients $\eta_i(\theta)$:

$$f(s,\theta) \approx \sum_{i=1}^{M} \sqrt{\lambda_i}\Phi_i(s)\eta_i(\theta) \tag{1}$$

where $\Phi_i(s)$ and λ_i are the eigenfunctions and the eigenvalues of the covariance function.

If a random field is Gaussian, then the coefficients η_i are independent standard Gaussian variables. In a general case their distribution must be identified. The multivariate distributions of the random projection coefficients (η_i) of the basis of each random field are characterized using the Polynomial Chaos Expansion (PCE, [6]). This technique consists in expanding the random coefficients in a sum of deterministic coefficients β_{ij} multiplied by a polynomial basis Ψ_j of multivariate germs $\boldsymbol{\xi}$ (whose probability density function is known) :

$$\eta_i(\theta) \approx \sum_{j=1}^{N} \beta_{ij}\Psi_j(\boldsymbol{\xi}(\theta)) \tag{2}$$

The germs distribution determines the polynomial family used as basis, which is orthogonal to the germs distribution. In this work Gaussian germs and so Hermite polynomials are employed. Finally, by generating ν germs $\boldsymbol{\xi}$ it is possible to obtain ν independent realisations of the generated field $f_{gen}(s)$ by combining KLE and PCE :

$$f_{gen}(s,\theta) = \sum_{i=1}^{M}\sum_{j=1}^{N} \sqrt{\lambda_i}\Phi_i(s)\beta_{ij}\Psi_j(\boldsymbol{\xi}(\theta)) \tag{3}$$

Some measured and generated realizations of the fields are shown in Fig. 1 and Fig. 2.

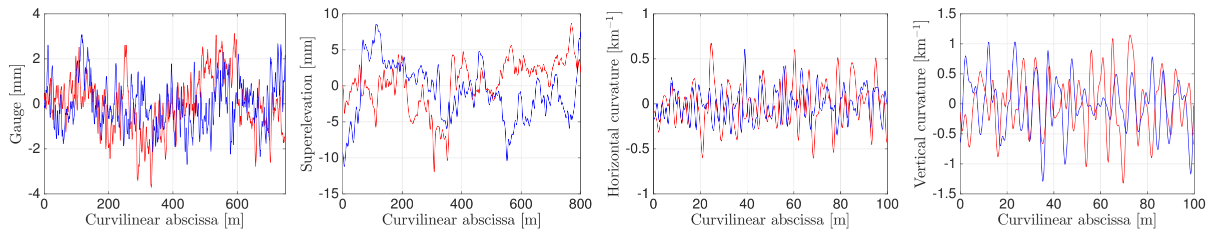

FIGURE 1 – Example of geometry irregularities, from left : gauge, superelevation, horizontal curvature and vertical curvature. Measurements (red line), generations (blue line).

The model is representative of the considered railway line : the spatial dependency and the probability density functions are captured. The matching between the measured and the generated tracks and the model validation through dynamical simulations are described in [1].

3 Deterministic numerical estimation of the fatigue criterion

The numerical computation of the rail fatigue criterion is decomposed into three calculations steps. The first one corresponds to the simulation of the passage of a train on a track affected by geometrical

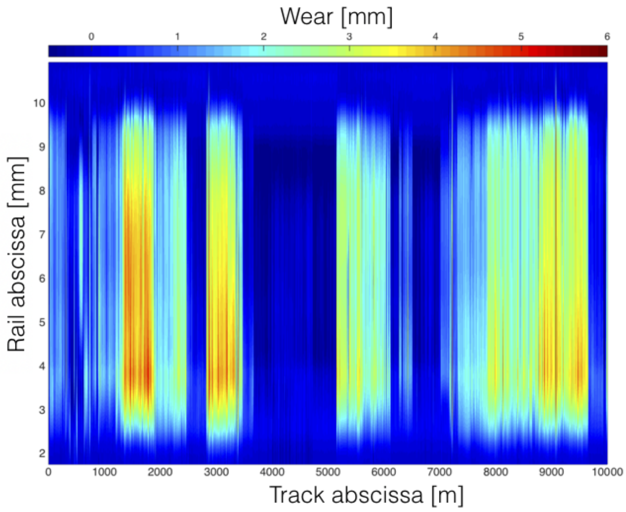

FIGURE 2 – Example of rail wear measurement.

irregularities. Then the stresses on the surface of the rail are used as boundary condition for a finite element computation of the rail internal stresses. At the end a multi-axial fatigue criterion is applied on these stresses.

3.1 Dynamical computation of the rail-wheel contact stresses

To characterize the wheel-rail contact patch in a point of the track, an advanced semi-Hertzian contact model [7] is used in the simulations. In this model the rail profile is discretized in strips along the lateral direction. Then, in each strip, the contact problem is solved as Hertzian. In order to take into account the rail plasticity, the contact stresses are limited and the shape of the contact zone is slightly modified [2].

Using the formulation described above, the contact stresses entirely depend on the wheel-rail relative position and velocity in each strip and the rail profile geometry. Since the friction coefficient is constant and the wheel is perfect solid of revolution (no wheel defects are considered), the wheel rotational angle and the track curvilinear abscissa are not involved. At this stage, the rails and the wheelset are considered as rigid bodies.

3.2 Rail internal stresses computation

The surface wheel-rail contact stresses, computed at a given track position, are used as boundary condition of a Finite Element computation of the internal rail stresses resulting from the passage of the wheel [8].

Two main hypotheses are made :

- The surface contact stresses field is stationary and is moving on the rail at constant velocity. A load cycle corresponds to a passage of this stress field on the rail.
- The rail is infinitely long along the longitudinal direction.

These hypotheses imply that each rail transversal section, is subjected to the same load cycle and the time variable is replaced, in the calculation, by the longitudinal spatial coordinate [8].

The rail material plasticity behaviour is considered by using a bilinear stress-strain law. At each passage of the load, the plastic strain tensor varies. After a different number of passages a steady state during repeated rolling-sliding is reached. This steady state corresponds to the condition according to which the plastic deformation does not change after another passage :

$$\int \dot{\bar{\bar{\varepsilon}}}_p(\mathbf{x}, s_i)\mathrm{d}x_1 = 0 \tag{4}$$

where $\bar{\bar{\varepsilon}}_p(\mathbf{x}, s_i)$ is the plastic strain tensor, $\mathbf{x} \in \mathcal{D}$.

The internal stresses in the rail transversal section corresponding to the steady state, $\bar{\bar{\Sigma}}(\mathbf{x}, s_i)$, are therefore used for the computation of the fatigue criterion. For simplicity the longitudinal coordinate is replaced by the time t :

$$\overline{\overline{\Sigma}}(\mathbf{x}, s_i) = \overline{\overline{\Sigma}}(\mathbf{x}_a, t, s_i) \tag{5}$$

with $\mathbf{x}_a = [x_2, x_3] \in \mathcal{A}$ and $t = x_1$.

3.3 Dang Van fatigue criterion

The steady state stress cycle in the rail section $\overline{\overline{\Sigma}}(\mathbf{x}_a, t, s_i)$, computed in Section 3.2, is used for the application of the Dang Van fatigue criterion [3]. This is a multi-axial fatigue criterion based on the passage from the macroscopic to the mesoscopic scale. When the material is isotropic, the mesoscopic stress tensor $\overline{\overline{\sigma}}(\mathbf{x}_a, t, s_i)$ is calculated as :

$$\overline{\overline{\sigma}}(\mathbf{x}_a, t, s_i) = \overline{\overline{\Sigma}}(\mathbf{x}_a, t, s_i) - \overline{\overline{\rho}}(\mathbf{x}_a, s_i) \tag{6}$$

where $\overline{\overline{\Sigma}}(\mathbf{x}_a, t, s_i)$ is the macroscopic stress tensor and $\overline{\overline{\rho}}(\mathbf{x}_a, s_i)$ is the centre of the smallest hypersphere circumscribed to the macroscopic stress deviator, $\overline{\overline{s}}(\mathbf{x}_a, t)$, temporal cycle.

Then the maximal shear stress is calculated as :

$$\tau(\mathbf{x}_a, t, s_i) = \frac{1}{2}(\sigma_1(\mathbf{x}_a, t, s_i) - \sigma_3(\mathbf{x}_a, t, s_i)) \tag{7}$$

where $\sigma_1(\mathbf{x}_a, t)$ and $\sigma_3(\mathbf{x}_a, t)$ are respectively the highest and the lowest principal mesoscopic stresses.

Finally the fatigue index $I(\mathbf{x}_a, s_i)$ is defined as :

$$I(\mathbf{x}_a, s_i) = \max_t \left(\frac{\tau(\mathbf{x}_a, t, s_i)}{b - a h(\mathbf{x}_a, t, s_i)} \right) \tag{8}$$

where a and b are material constant (coming the torsion and traction fatigue tests) and $h(\mathbf{x}_a, t)$ is the mesoscopic hydrostatic pressure. If the index is bigger than 1, the structure is subject to fatigue. The biggest the index is, the less load cycles the structure can be subjected to.

4 Fatigue criterion stochastic model

4.1 Dynamical computation of the rail-wheel contact forces and kinematic parameters

Since the dynamical simulations are not heavily time-consuming (around 1 minute for a simulation of a 100 m track portion), the random space, i.e. the germs $\boldsymbol{\xi}(\theta)$ in Equation (3), is explored through Monte Carlo sampling (MCS), in which each simulation corresponds to a generated track geometry and rails wear. Let $\mathbf{u} = \mathbf{u}(s, \theta)$ be the multi-dimensional random fields representing the evolution of the rail-wheel kinematic parameters and the rail profile geometry, i.e. the output of the dynamical simulations. When all the input parameters of the dynamical simulations are stationary in the space, the field $\mathbf{u}$ is stationary, i.e. its statistics do not depend on the track curvilinear abscissa s :

$$p_\theta(\mathbf{u}(s_1 + s', \theta), \ldots, \mathbf{u}(s_n + s', \theta)) = p_\theta(\mathbf{u}(s_1, \theta), \ldots, \mathbf{u}(s_n, \theta)) \tag{9}$$

where $p_\theta(\mathbf{u}(s_1, \theta), \ldots, \mathbf{u}(s_n, \theta))$ is the marginal probability density function of order n of the random field $\mathbf{u}(s_1, \theta)$.

4.2 Construction of the polynomial chaos metamodel of the fatigue criterion

The computation of the rail internal steady state stress cycle, described in Section 3.2, is the most expensive stage. A simulation lasts from 30 to 80 minutes, making the Monte Carlo sampling not advisable for exploring the random space. For this reason the computation of the internal load cycle and the application of the Dang Van fatigue index on them (Section 3.3) is replaced by a metamodel. After its construction the metamodel allows the exploration of the random space without computational effort. In this section the polynomial chaos based metamodel of the fatigue index is presented.

The quantity of interest at this stage is the random fatigue index defined in Eqn. (8) in the spatial domain : $I(\mathbf{x}_a, s, \theta)$.

4.2.1 Input random space of the metamodel

Since the mechanical properties and the structure geometry of the rail are considered as deterministic, the load cycle in the rail (and so the fatigue index) only depends on the boundary condition, i.e. the surface wheel-rail contact stresses. Because of the contact model described in Section 3.1 , the contact stresses field is entirely determined by the kinematic parameters listed in Section 3.1 and grouped in the stationary random field $\mathbf{u}(s,\theta)$ introduced in Section 4.1, that constitute the input random space of the metamodel. The statistics of this random field are estimated by Monte Carlo sampling, in which each simulation corresponds to a generated track irregularities field.

The PCE metamodel of the fatigue index is equal to :

$$I(\mathbf{x}_a,s,\theta) \approx \sum_{k=1}^{T} \alpha_k(\mathbf{x}_a)\zeta_k(\mathbf{u}(s,\theta),s) \tag{10}$$

4.2.2 Polynomial Chaos Expansion basis

The PCE basis has to be orthogonal with respect to the random space [9] :

$$\int_{\Theta} \zeta_k(\mathbf{u}(s,\theta),s)\zeta_l(\mathbf{u}(s,\theta),s)p_\theta(\mathbf{u}(s,\theta))\mathrm{d}\theta =< \zeta_k(\mathbf{u}(s,\theta),s),\zeta_l(\mathbf{u}(s,\theta),s) >= \delta_{kl} \tag{11}$$

where δ_{kl} indicates the Kronecker delta.

Because of the stationarity of $\mathbf{u}(s,\theta)$, Eqn. (9), the orthogonality condition does not depends on the curvilinear abscissa s :

$$\zeta_k(\mathbf{u}(s,\theta),s) = \zeta_k(\mathbf{u}(s,\theta)) \quad \forall s \in [0,S] \tag{12}$$

As proposed in [10], the PCE basis, orthogonal with respect to an arbitrary multi-dimensional random space, can be constructed by Gram-Schmidt orthogonalization method.

4.2.3 Polynomial Chaos Expansion coefficients

The PCE coefficients $\{\alpha_k\}$ in Eqn. 10 are computed by least-square regression [4]. These approach minimizes the mean squared error between the prediction and the solution.

Let $\{\mathbf{u}(s_1,\theta_1),\ldots,\mathbf{u}(s_R,\theta_R)\}$ be the set collecting R regression points. This set is obtained by Monte Carlo sampling R realizations of the field $\mathbf{u}(s,\theta)$ at curvilinear abscissa s randomly chosen in the interval $[0,S]$. Because of the stationarity of $\mathbf{u}(s,\theta)$, the choice of s is completely arbitrary.

Let ζ denote the matrix whose the coefficients are given by $\zeta_{ij} = \zeta_j(\mathbf{u}(s_i,\theta_i))$ with $i = 1,\ldots,R$ and $j = 0,\ldots,T$. Let $\mathcal{I}$ the vector containing the fatigue indices, Eqn. 8, computed in the regression points : $\mathcal{I}_k = I(\mathbf{x}_a,s_k,\theta_k)$ with $k = 1,\ldots,R$. The PCE coefficients are calculated as :

$$\mathbf{A} = (\zeta^{\mathrm{T}}\zeta)^{-1}\zeta^{\mathrm{T}}\mathcal{I} \tag{13}$$

where $\mathbf{A}$ is a vector such that $\mathbf{A}_k = \alpha_k(\mathbf{x}_a)$.

As described in [11], the use of a sparse PCE is more efficient than a complete basis PCE. The idea of the sparse PCE it to use only some polynomials functions instead of the complete basis. In this paper the PCE coefficients are firstly computed with a complete basis as described below. The 0-degree polynomial term is always selected in the sparse basis because it determines the mean value of the quantity of interest. Then, the PCE coefficients $\alpha_k(\mathbf{x}_a)$, with $k > 1$, are sorted according the descending order of their squared value. Let $\widetilde{\alpha}_k(\mathbf{x}_a)$ be the sorted PCE coefficients. The sparse PCE basis is obtained by selecting the 0-degree polynomial term and $T'(\mathbf{x}_a)$ more terms such that $T'(\mathbf{x}_a)$ is the minimal number satisfying the following condition :

$$\frac{\sum_{k=1}^{T'(\mathbf{x}_a)} \widetilde{\alpha}_k(\mathbf{x}_a)^2}{\sum_{k=1}^{T} \alpha_k(\mathbf{x}_a)^2} \geq 1-\varepsilon_s \tag{14}$$

where ε_s is a tolerance. Then the PCE coefficients are recomputed, as in Eqn. 13, by taking only the $T'(\mathbf{x}_a)$ terms satisfying the sparsity criterion and the 0-degree polynomial term. Let $\widehat{\alpha}_k(\mathbf{x}_a)$ and $\mathcal{K}(\mathbf{x}_a)$ indicate respectively the recomputed coefficients and the sparse indices set.

The metamodel error is evaluated by Leave-One-Out (LOO) cross-validation. Supposing that the metamodel would have been constructed with all the regression points but the i-th one, the difference between the deterministic solution and the metamodel prediction in the i-th point can be calculated. Repeating this step in all the R regression points, the squared mean of these differences is the LOO error. As shown in [12], the relative LOO error, $\varepsilon_l(\mathbf{x}_a)$, can be analytically calculated as :

$$\varepsilon_l(\mathbf{x}_a) = \frac{1}{R}\sum_{i=1}^{R}\left(\frac{I(\mathbf{x}_a, s_i, \theta_i) - \sum_{k\in\mathcal{K}(\mathbf{x}_a)} \widehat{\alpha}_k(\mathbf{x}_a)\zeta_k(\mathbf{u}(s_i, \theta_i))}{(1-l_i)\mathrm{Var}(I(\mathbf{x}_a, s, \theta))}\right)^2 \tag{15}$$

where l_i are the diagonal terms of the matrix $\widehat{\zeta}(\widehat{\zeta}\widehat{\zeta}^{\mathrm{T}})^{-1}\widehat{\zeta}^{\mathrm{T}}$ with $\widehat{\zeta}$ being the matrix constituted by the columns of ζ corresponding to the sparse indices set $\mathcal{K}(\mathbf{x}_a)$.

5 Conclusion

The steps to lead to a stochastic characterisation of the track geometry irregularities and the rail wear of a railway network are presented in this work. A large set of experimental measurements is used to model the irregularities as random fields. The model is able to capture the statistical and the spatial variability of the irregularities on the considered track.

A stochastic model of the rail fatigue, that considers the track irregularities and the rail wear as random fields, is constructed by means of numerical simulations. A PCE metamodelling technique allows to reduce the number of expensive numerical simulations needed to build the random model.

This rail fatigue model can be used to improve the railway network maintenance. Indeed, a sensitivity analysis study can be performed by means of this model in order to identify which kind of irregularities affects the rail life.

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