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Taylor Meshless Method applied to composite plates

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Abstract — This paper introduces a new meshless method using Taylor series to deduce the shape functions. The point-collocation is discretized only on the boundary without integration. Coupled with least-squares method, one can increase the degree of polynomials easily while the computation doesn't increase too much. In this paper, the techniques and advantages of this method are illustrated with some examples, including Kirchhoff plate and composite plates problems.
Key words — Meshless method, Taylor series, Computational mechanics.

1. Introduction

The numerical simulation is an indispensable step in manufacturing processes. It can help to optimize the products and shorten the manufacturing cycle. The key of the simulation is to obtain the solution of partial differential equations for a certain problem. Meshless methods have been widely developed in recent decades. Some monographs have been published. There are a number of meshless methods, such as the moving least-squares (MLS) [8], the reproducing kernel particle method (RKPM) [13], the partition of unity method (PUM) [5], the point interpolation method (PIM) [3] and Natural neighbor interpolation [7, 9, 12].

The "Taylor Meshless Method" (TMM) solves the partial differential equations in their strong form in the area without any background mesh. Only the collocation points on the boundary are needed, which means it is a true "meshless method" and can be applied to solve problems with variable coefficients and nonlinear problems.[4]. The number of unknowns can be reduced significantly using the technique of Taylor series. With the least-squares collocation, the errors converge exponentially with the degree of polynomials[1, 2, 11]. The aim of the paper is to apply TMM to plate bending and discuss the robustness and convergence of this method.

This paper is organized as follows. In Section 2, we presents the techniques of TMM with the examples of Kirchhoff plate and discuss its efficiency and accuracy. In Section 3, we consider rectangular bidirectional composites and sandwich plates to discuss the behavior of TMM.

2. Application of TMM to Kirchhoff plate problem

$$\begin{cases} \dfrac{\partial^2 M_{\alpha\beta}}{\partial x_\alpha \partial x_\beta} + p = 0 \\ k_{\alpha\beta} = -\dfrac{\partial^2 w}{\partial x_\alpha \partial x_\beta} \\ \{M\} = [D]\{K\} \\ w(\underline{x}) = w^d(\underline{x}) \quad \text{on } \Omega^d \\ w(\underline{x}) = w^n(\underline{x}) \quad \text{on } \Omega^n \end{cases} \tag{1}$$

Eq.(1) has three equations in second order derivatives with a solution $w(x,y)$, $k_x(x,y)$ for plates in the Kirchhoff plate theory. The approximate solution can be expanded with complete polynomials of degree *N*:

$$\begin{cases} w(x,y)=\sum_{m=0}^{N}\sum_{n=0}^{N-m} w(m,n)(x-x_0)^m (y-y_0)^n \\ k_x(x,y)=\sum_{m=0}^{N}\sum_{n=0}^{N-m} k_x(m,n)(x-x_0)^m (y-y_0)^n \end{cases} \tag{2}$$

With the technique of TMM, one can get derive other coefficients of $w, k_x, \bar{k}, M_x, \bar{M}$ with the initial data $w(0,n), w(1,n), k_x(0,n), k_x(1,n)$. Table 1 gives the recurrence formulae for the elements of TMM.

Table 1 – Organization of the computation

Equation	Derivation
$k_x=-\dfrac{\partial^2 w}{\partial x^2}$	(1) $w(m,n)=\dfrac{1}{m(m-1)}k_x(m-2,n)$
$\bar{k}=\begin{Bmatrix} k_y \\ k_{xy}\end{Bmatrix}=-\begin{Bmatrix} \dfrac{\partial^2 w}{\partial y^2} \\ \dfrac{\partial^2 w}{\partial x\partial y}\end{Bmatrix}$	(2) $\bar{k}(m,n)=-\begin{Bmatrix} (n+2)(n+1)w(m,n+2) \\ (m+1)(n+1)w(m+1,n+1)\end{Bmatrix}$
$\bar{M}=\begin{Bmatrix} M_y \\ M_{xy}\end{Bmatrix}=\begin{Bmatrix} D_{12} \\ D_{13}\end{Bmatrix}k_x+\bar{D}.\bar{k}$	(3) $\bar{M}(m,n)=\begin{Bmatrix} D_{12} \\ D_{13}\end{Bmatrix}k_x(m,n)+\bar{D}.\bar{k}(m,n)$
$\dfrac{\partial^2 M_x}{\partial x^2}=-2\dfrac{\partial^2 M_{xy}}{\partial x\partial y}-\dfrac{\partial^2 M_y}{\partial y^2}-p$	(4) $M_x(m,n)=\dfrac{-1}{m(m-1)}\begin{Bmatrix} p(m-2,n)+2(m-1)(n+1)M_{xy}(m-1,n+1) \\ +(n+2)(n+1)M_y(m-2,n+2)\end{Bmatrix}$
$k_x=\dfrac{1}{D_{11}}\{M_x-D_{12}k_y-D_{13}k_{xy}\}$	(5) $k_x(m,n)=\dfrac{1}{D_{11}}\{M_x(m,n)-D_{12}k_y(m,n)-D_{13}k_{xy}(m,n)\}$

First, we set *p* to 0. The shape functions can be obtained as $P_j, 1\le j\le 4N+2$ which means the degrees of freedom are 4*N*+2. Second, one can find a particular solution $w_s=P_s$ for Eq.(1). The approximate solution for the problem becomes: $w=P_s+\sum_{j=1}^{4N+2}P_j\alpha_j$. To determine the vector α, a set of points on the boundary of the domain are chosen. M_1 points are chosen on Ω^d and M_2 points are chosen on Ω^n . A least-squares method is used to minimize the function:

$$J(\alpha)=\frac{1}{2}\sum_{j=1}^{M_1}\left\| P_j\{\alpha\}+(P_s)_j-w_j^d\right\|^2+\frac{1}{2}\sum_{j=1}^{M_2}\left\|\frac{\partial P_j}{\partial n}\{\alpha\}+\frac{\partial (P_s)_j}{\partial n}-w_j^n\right\|^2 \tag{3}$$

2.1. Resolution of Kirchhoff plate problem with a circular domain

Now we consider a circular Kirchhoff plate with clamped edges. The problem can be described as:

$$
\begin{cases} D_0 \nabla^4 w = p \\ w(a) = 0 \\ \partial w / \partial n \big|_{r=a} = 0 \end{cases} \tag{4}
$$

where a is the radius of the plate. With these boundary conditions, the theoretical solution is $w = \left(a^2 - r^2\right)^2 p \big/ 64 D_0$ [10].

Figure 1 is the convergence of TMM along with the increase of the collocation points when N=10, 20 and 30. When the number of collocation points is large enough, the solution is stable with a high accuracy. To ensure the convergence, one can choose 4p points according to the degrees of freedom.

Figure 2 is the convergence of TMM along with the increase of the degree of the polynomial. The solution will be very accurate if the theoretical solution is a polynomial. When the degree is more than 5, the solution becomes less accurate due to the propagation of round-off errors. Nevertheless, the accuracy remains 10^{-11} up to a high degree.

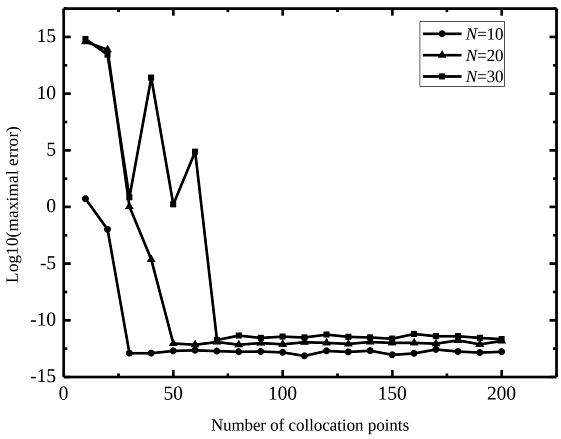

Figure 1 –The influence of the number of collocation points on the convergence for circular Kirchhoff plate

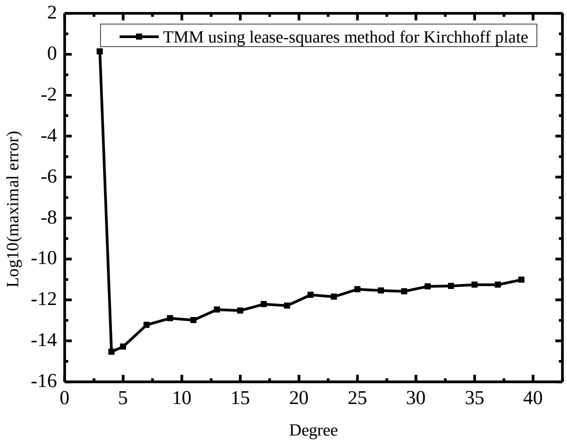

Figure 2 –The convergence of TMM for circular Kirchhoff plate

2.2. Resolution of Kirchhoff plate problem with a rectangular domain

The domain is a square with the width 1. One chooses a uniformly distributed cloud.

1) Rectangular plate, simply supported, uniform load

For a rectangular plate, when it is simply supported, the problem can be described as:

$$\begin{cases} D_0 \nabla^4 w = p \\ w = 0 \qquad \text{on } \partial\Omega^d \\ \partial^2 w / \partial n^2 = 0 \qquad \text{on } \partial\Omega^n \end{cases} \tag{5}$$

One can find an approximate solution for the problem in [16].

Figure 3 is the convergence of TMM along with the increase of the degree of the polynomial. One can find an exponential convergence as in many other examples. The propagation of round-off errors has some influence on the accuracy after degree 20. Nevertheless, the accuracy stays around 10^{-3} with a high degree.

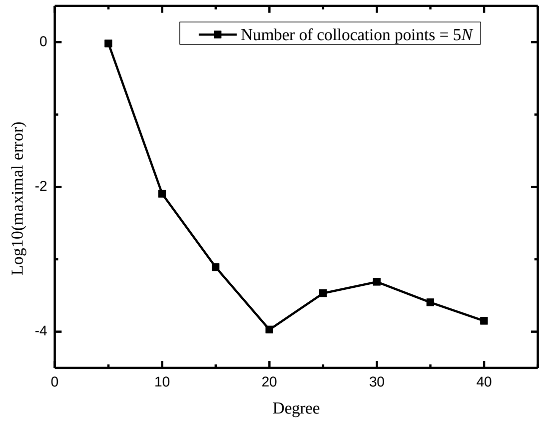

Figure 3 – The convergence of TMM for rectangular Kirchhoff plate

2) Rectangular plate, clamped edges, uniform load

A rectangular Kirchhoff plate with clamped edges and uniform load can be described as:

$$\begin{cases} D_0 \nabla^4 w = p \\ w = 0 \qquad \text{on } \partial\Omega^d \\ \dfrac{\partial w}{\partial n} = 0 \qquad \text{on } \partial\Omega^n \end{cases} \tag{6}$$

Table 2 shows the results of the plate center with different methods. Theoretically, M_x should equal to M_y. One can find that the accuracy of TMM is at least on the same level as with other methods.

Table 2 – Deflection and moment of the centre of a rectangular plate, clamped edges, uniform load

Method	Uniform load p		
	w_{max}(×D_0/pa^4)	M_x(×1/pa^4)	M_y(×1/pa^4)
TMM (Degree 20)	0.0012653	0.022904	0.022906
MLPG (Meshless Local Petrov-Galerkin)[14]	0.001258	0.002288	0.002288
Boundary Element Method[6]	0.001255	0.02282	0.02282
Theoretical Solution[10]	0.001260	0.02310	0.02310

3. Application of TMM to rectangular bidirectional composites and sandwich plates

Distributed loading:

$$q_0(x,y) = \sigma \sin px \sin qy \tag{7}$$

where $p = p(n) = n\pi/a, q = q(m) = m\pi/b(n, m = 1, 2, 3\cdots)$, σ is a constant. We let m=n=1, a=b=1, $\sigma = 1$. The governing equation can be simplified as:

$$D_{11}w_{,xxxx} + (D_{12} + D_{21} + 4D_{66})w_{,xxyy} + D_{22}w_{,yyyy} = q_0 \tag{8}$$

The exact solution for this problem is: $w_e = C \sin px \sin qy$, where C is a constant. It is determined by bending stiffness D_{ij} and the loading q_0. The layer material coefficients are in [11]. The plate is a square sandwich plate under the distributed loading. The materials of three layers are the same. Figure 4 shows the fiber directions of each layers.

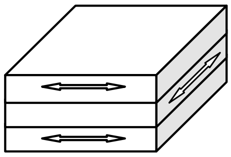

Figure 4 – Fiber directions and thickness of each case

The Taylor series are expanded at the center of the plate. The convergence with the degree is illustrated in Figure 5. One can get an exponential convergence by increasing the degree up to 16. Next, the results become less accurate due to the round-off errors. The accuracy of normal stresses at the centre of the plate stands around 10^{-4}. The accuracy of shear stresses is not good enough because the points we test are far from the expand point. This problem can be solved by the technique of sub-domains.

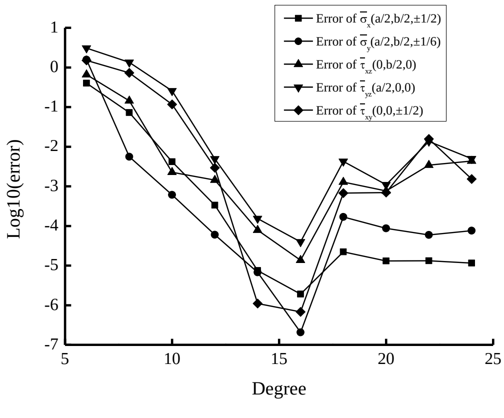

Figure 5 – The influence of the degree on the convergence. Number of collocation points = 6N.

4. Conclusions

2D elastic problems have been discussed to illustrate the efficiency and accuracy of TMM. We obtain exponential convergence with rather few degrees of freedom and number of collocation points. In this paper, TMM has been restricted to 2D problems. However, it can be used to solve 3D nonlinear problems combined with linearization methods.

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